Residential equilibrium
in a multifractal metropolitan area

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Abstract

We combine a residential location model derived from urban economics, with the geometry of a multifractal Sierpinski carpet in order to represent and model a metropolitan area. This area is made on the one hand of an urban system hierarchically organize around a city-center, and on the other hand, of green areas arranged in an inverse hierarchal order. Households seek for a diversified basket of commodities. An analytical solution is obtained by using an specific geographic coding system for computing distances. The values of the parameters used in the model are inspired from the French urban reality; realistic basic solutions are proposed and comparative static analyses simulations are performed. The results show that the well-known French periurbanisation process (1970 onward) can be explained by an increase in income and a reduction in transportation costs. Nevertheless, changes in preferences characterized by a increased taste for open spaces can also contribute to urban sprawl by making the gradient of land rent less steep and by generating more peripheral households locations.

Key words: periurban, residential localization, fractal geometry, amenities
Classification JEL: R12, R21

1. Introduction

Most metropolitan areas in developed countries are spread out over a large area comprising a patchwork of, on the one hand, productive business parks and housing which provides the population with jobs, amenities and urban public goods, and, on the other hand, open spaces, agricultural and forest-covered green areas which offer a living environment close to nature with landscapes and recreational green amenities. We here propose a model of residential location in the urban economics tradition combined with a geometric fractal model where space, made up of urban sites and green areas, is heterogeneous. We chose sufficiently realistic parameters for the model in order to simulate the effects of various impacts: income, transportation costs, changing taste of households for amenities, etc.

Numerous empirical works have studied the role of "green" amenities in suburban and large metropolitan areas, particularly by estimating their hedonistic prices (see e.g. Bender *et al*., 1997; Bolitzer *et al*. 2000; Cheshire and Sheppard, 1995; Geoghegan *et al*., 1997; Hobden *et al*., 2004; Irwin, 2002; Mooney and Eisgruber, 2001; Paterson and Boyle, 2002; Roe *et al*., 2004; Thorsnes, 2002; Tyrvainen and Miettinen, 2000). From an analytical point of view, studies in this domain are quite rare. Recent microeconomic models of urban economics which focus on amenities have recently been formulated (Brueckner *et al*., 1999; Marshall, 2004; Turner 2004; Wu and Plantinga, 2003), some of them being calibrated or estimated on structural equations (Bates and Santerre, 2001; Cavailhès *et al*., 2004a; Cheshire and

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Sheppard, 2002). However, the geometry of these models is that of a Thunian space with interlocking rings; it is little adapted for modeling real world settlements, which are made up of heterogeneous objects. Cavailhès *et al*. (2004b) have offered a microeconomic program of residential localization based on the fractal pattern of a Sierpinski carpet. The results differ from those of urban economy such as those of von Thünen, particularly because land rent gradients are not monotonous in the distance from the central business district (CBD).

This paper distances itself from the most restrictive hypotheses of Cavailhès *et al*. (2004b): on one hand, the size of a residential plot here depends on the price of the land, which offers the household a trade off between cost of land and transportation. On the other hand, because our Sierpinski carpet is multifractal, we can obtain residential sites larger in the centre of the urban hierarchy than in the periphery. This analytical model allows us to simulate the effects of changes in the economic parameters or in household preferences.

The first section includes a presentation of the analytical models in which we develop the geometric model of the multifractal Sierpinski carpet, the microeconomic model of residential location and, finally, the spatial model (coding the coordinates of sites and computing distances). The results of a benchmark solution with parameters close to the real world are presented in Section 2. Simulations of comparative statics are presented in Section 3 which enables us to discuss the properties of this economic-fractal representation. Section 4 is devoted to our conclusions.

2. Model

2.1. Geometric model of the multifractal Sierpinski carpet

2.1.1 The spatial model

Here we attempt to analyze the structure known in France as an *aire urbaine* which is close to the *Metropolitan Statistical Areas* in the U.S.A.: a central city (where we suppose employment is concentrated) surrounded by suburban cities and periurban towns and villages, with open spaces and green areas of differing sizes filling in the vacuum left by the urban site complex. The size of the built-up sites and the green areas in our stylized representation depends on their position in the fractal hierarchy of the Sierpinski carpet.

Fractal geometry is frequently used for describing urban patterns or/and for verifying if urban patterns follow fractal laws (see e.g. Arlinghaus and Arlinghaus, 1989; MacLennan *et al*., 1991; White and Engelen, 1993 and 1994; Batty and Longley, 1994; Batty and Xie, 1996; Frankhauser, 1994 and 1998; De Keersmaecker *et al.*, 2003 or Thomas *et al*., 2006), as well as for describing urban dynamics in using for example, cellular automata (see e.g. Batty, 1991; Batty and Longley, 1994; Bailly, 1999). Some of these works have economic foundations (White and Engelen, 1993; 1994), but their approaches are different from ours: they are mainly inductive (by the estimation of a fractal dimension based on the real world, for example) while our approach is hypothetico-deductive. Unlike the above-mentioned papers, we here deal with a high level of abstraction, the equivalent, for example, of the hexagonal market areas of Christaller, or the concentric von Thünian rings. By doing so, we do not aim at modeling realistically one specific urban area, but at representing an abstract urban area using the most realistic values for the parameters. Indeed, if one considers the metropolitan patterns as being made of two types of objects (built-up sites and green areas), it would be unrealistic to explain the complexity of the observed patterns based only on three equations (size of the plots, land rent and population). Hence, a calibration extends beyond the scope of this paper. It is possible and hence preferable, to examine the properties of the model with realistic parameters rather than using arbitrary values.

The usual Sierpinski carpet is a regular fractal which is constructed following an iterated application. We opted for an *outer mapping procedure*[1]. We begin with the first residential site with a square shape (Figure 1). The first iteration allows us to obtain the « generator » of the carpet. It is created by placing four other sites, whose size is reduced by a factor $a < 1$ (what we call "satellite sites"), in the corners of the initial site. The generator is reduced by the factor a with the following iteration, and four of these smaller replicas are again placed in the corners of the figure obtained during the precedent step which gives four new urban centers, each of them surrounded by four satellites. This procedure is applied to each step. Figure 2 shows the result of the third step.

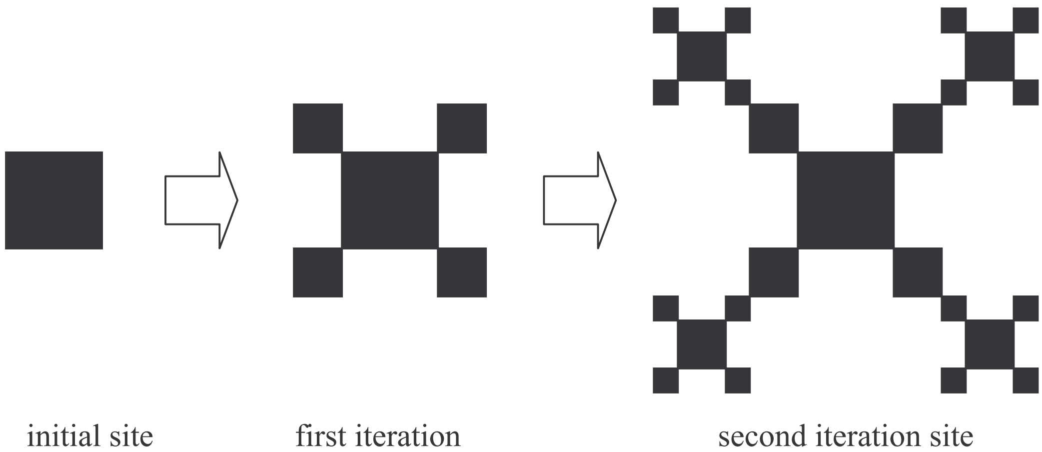


Figure 1: Multifractal Sierpinski carpet with the first two iterations

We assume that the square-like elements of different size correspond to the residential areas which are all connected following the logic of the Sierpinski carpet. The transportation network that is used is integrated in these residential sites, trips are made following the shortest itinerary. The non-occupied sites are interpreted as green areas.

2.2.2 Coding the sites of the spatial model

We introduce now a coding system according to the logic of iteration (Figure 2). The code is composed of three digits $C_3C_2C_1$ for each urban site. The digit on the right, C_1, takes the value of $C_1 = 0$ when it represents a site surrounded by four satellite sites (therefore, we use the term "urban center"). The four satellite sites which surround it take the values of C_1 which respect the symmetrical aspect of the structure. The value $C_1 = 1$ is attributed to the site closest to the urban center with an immediately superior rank in the hierarchy, the value $C_1 = 2$ corresponds to the two sites which are on the lateral branches, and the value $C_1 = 3$ is reserved for sites situated on the principal axes opposite sites $C_1 = 1$. Thus, for example, as is demonstrated by figure 3, site (141) is closer to the center (100) than the two sites (122) and (123), just as site (111) is closer to the center (100) than sites (112) or (113). We find the

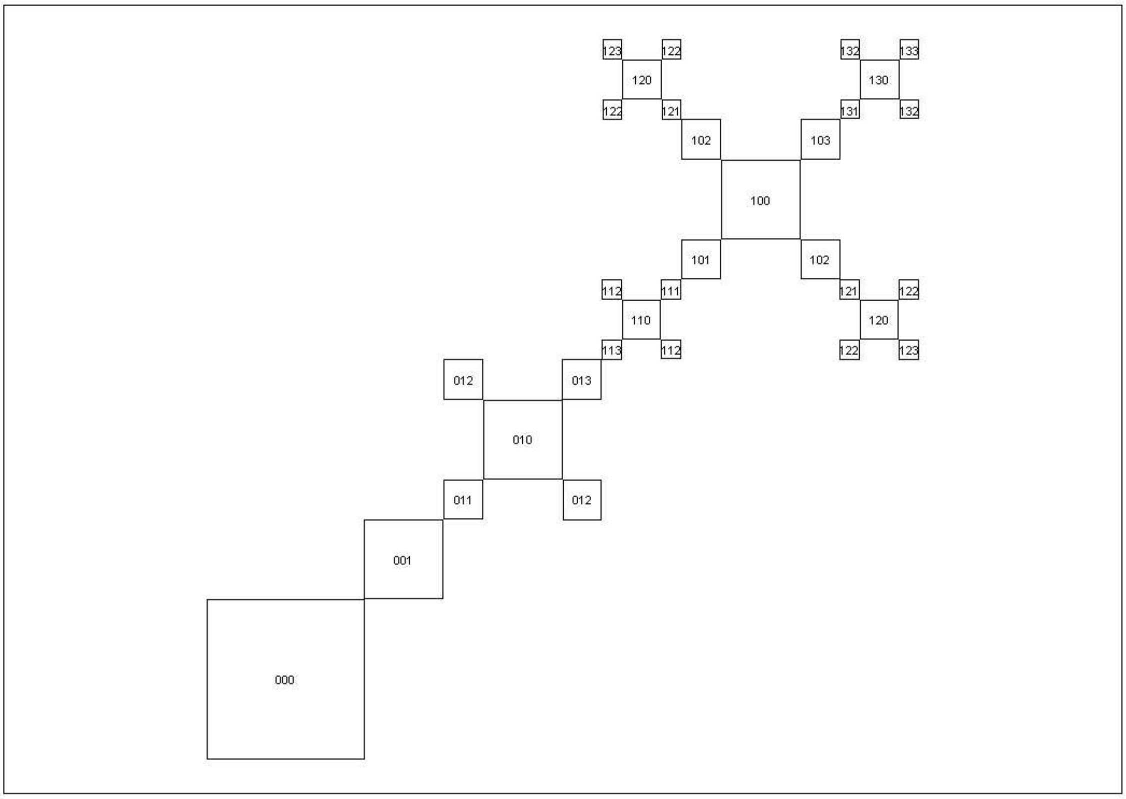
[1] The more current *inner mapping procedure* is not adapted to modeling a metropolitan area. Effectively, since the iterations are done by the fragmentation of the initial site, this procedure would lead to a surface of the urban sites that would shrink in course of iterations.

same logic for the following hierarchical level: site (110) and the associated sites (111) to (113) are closer to the principal center (000) than sites (120). On the other hand, the principle of symmetry results in giving the same value $C_1 = 1$ to the four satellite sites of the generator which surround site (000). Thus, in general, sites which have the same accessibility are designated by the same code.

A second digit C_2 is placed before the digit $C_{1,}$ indicating the position of centrality in a set of 25 elements in the carpet. Thus, site (100) is surrounded by 24 sites: by its own satellites which are designated as (101) to (103), then by four urban sites whose codes are (110), (120) and (130), who have their own satellites (codes (111) to (113), etc.). We note here, as we did for figure 3, that the digit $C_1 = 1$ is once again attributed to the site closest to the next urban center in the hierarchy: site (110) and the associated sites (111) to (113) are closer to the principal center (000) than sites (120) and (130) and their associated sites.

For a Sierpinski carpet developed up to any stage K, each site is then characterized by a code $C_K\ C_{K-1}\ C_{K-2} \ldots\ C_j \ldots C_2\ C_1$, which is determined by generalizing the principle which has just been shown.

Figure 2. Coding of residential sites (northeastern part of the Sierpinski carpet)



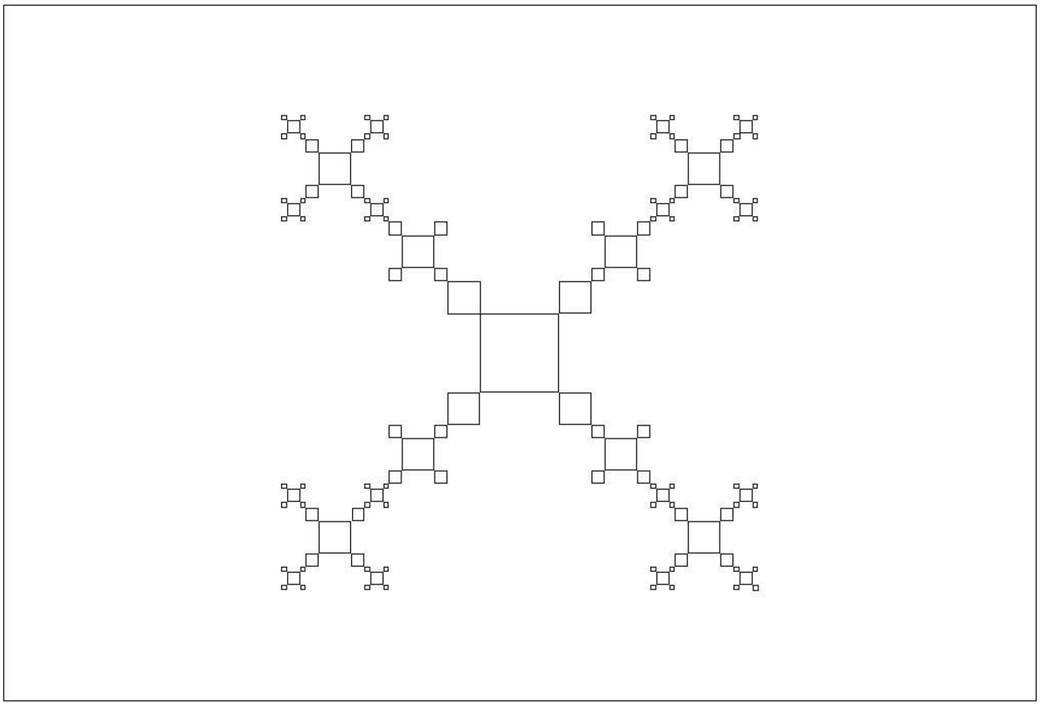
This coding system facilitates the computation of the distances between the different urban areas and the green areas.

We now introduce the localization of the different types of economic activities. We assume that employment is concentrated in the central site (000) which is the destination for all the workers. These workers represent the impersonal households who primarily consume the amenities and urban public goods offered in the residential sites. These differentiated products, in a geographical economic sense (Fujita and Thisse, 2002; Krugman, 1991), are arranged according to a hierarchy. Following Christaller's logic, we assume that the hierarchical differentiation of these urban products depends on the hierarchy of the Sierpinski carpet: those of a superior rank, which are the least often consumed, are exclusively offered in the central site (000) and we call this an urban centre of order K, then the four urban sites (100) correspond to the order $k = K - 1$, they offer the amenities of rank $K - 1$, etc. The lowest level ($k = 0$) offers only local public goods. We therefore assume that a localization in a residential site of rank k provides the urban amenities of rank $\leq k$. We emphasize that sites of same order don't have all the same size which depends on the distance of the main center (000). Hence the (010) is larger than (110) although they are both of order $k = 1$.

The consumer also seeks "rural" amenities and recreational services offered by the green areas which are found in the interstices between residential sites. These are also differentiated products and hierarchically arranged: for each site k' of green area, a specific rural amenity k' is offered, according to its hierarchical level in the Sierpinski carpet. Hence each residential area is surrounded by local green areas of order $k' = 0$, which are smaller than the areas lying farer away of higher order (figure…). Here, too, the size of the green areas of a given order k depends on their localization within the pattern.

We also hypothesize a consumer's taste for variety which is a classic in economy. The consumer then goes to different urban and green areas in order to obtain various amenities and public goods. This involves a transportation cost which is added to the alternating migrations of home-work (commuting is not multi-objective).

Figure 3. Multifractal Sierpinski carpet with the third iteration

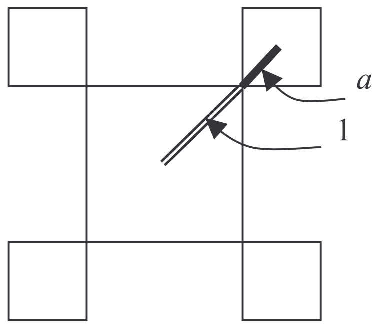


2.2.3 Computing distances

Our aim is now to calculate formulas allowing to compute the distances from the different residential areas to the centre of order *k*. We assume again that iteration has been done up to the third step. We designate the distance between a site ($C_3C_2C_1$) and a site ($C_3'C_2'C_1'$) by $d(C_3C_2C_1, C_3'C_2'C_1')$. If ($C_3'C_2'C_1'$) is the code of the centre of order *k* we write alternatively $d_k(C_iC_jC_k,) \equiv d(C_iC_jC_k, C_3'C_2'C_1')$. This is possible since we know which one of the centers of order *k* corresponds to site ($C_iC_jC_k$,). The computation of the distances which separate these points must take into account that, in a multifractal Sierpinski carpet, the sites can be of different sizes. [2]

Firstly, we take consideration of the generator (figure 4). We take the half-diagonal of the initial site as the unit of distances without loss of generality. At the first iteration, the half-diagonal of the four satellite sites of the generator is reduced by the factor *a*. In the same way, at each following iteration, smaller sites are generated whose half-diagonals are multiplied each time by the factor[3] *a*.

Figure 4. Measuring the distances



In order to compute distances, these are decomposed into *partial distances* following a principle demonstrated by the example of the itinerary which links one of the sites (132) to the central site (000), that is $d(132, 000) \equiv d_3(132)$ (cf. Figure 3). Sites (132) are generated during the third step. Their main urban center is then site (100) which is directly linked to the center (000). Moreover, sites (132) are direct satellites of site (130) which concentrate all the incoming flow from sites (131), (132), (133). Thus, we obtain:

$$d_3(132) \equiv d(132, 000) = d(100, 000) + d(130, 100) + d(132, 130)$$
$$\equiv d^{(3)}(100) + d^{(2)}(130) + d^{(1)}(132)$$

where we have introduced the *partial distances d* whose index indicates that it refers to the distance to the center of the closest order *k*. One thus has:

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[2] For technical reasons, we must introduce a small intra-site distance; this does not affect the simulation results.

[3] In order to guarantee that for all residential areas the distances to the nearest center of order *k* corresponds really to that is one we expect in concordance with the coding i.e. the logic of the Sierpinski carpet , we should assume that $a \geq 0.5$. Indeed this would no longer be the case e.g. for the distance (101) to (100) with respect to the distance (101) to (110). Hence for goods of order k =1 consumers would prefer to go to (110).

$$d^{(3)}(C_3 00) = d(C_3 00,\ 000)$$
$$d^{(2)}(C_3 C_2 0) = d(C_3 C_2 0,\ C_3 00)$$
$$d^{(1)}(C_3 C_2 C_1) = d(C_3 C_2 C_1, C_3 C_2 0)$$

One then obtains in a general way:

$$d(C_3 C_2 C_1,\ 000) = d^{(3)}(C_3 00) + d^{(2)}(C_3 C_2 0) + d^{(1)}(C_3 C_2 C_1)$$

The *partial distances* can be written with the general formula as follows:

$$d_k(C_3 C_2 C_1) = \zeta^{(k)} a^{l(k)}\ (1+a)^{m(k)}\ (1+2\,a)^{n(k)}$$

where the exponents $l(k),\ m(k),\ n(k)$ depend on codes $C_3C_2C_1$ and on the order k. The factor $\zeta^{(k)}$ is an operator whose values depend on order k, which makes it possible to take into account that which does not necessarily always have to pass by the assigned centre of a higher rank: as a result, in order to go to site (000) from sites (111), (110), (113), etc., one does not pass by site (100).

This factor has a constant value of $\zeta^{(3)} = (1 - \delta_{C3,0})$ for $k = 3$. For the other k-values, the operator can take the values –1, 0 and +1. By introducing the value 0, one can eliminate the terms which are not present for certain sites: for example, the distances $d^{(3)}(C_3 00)$ and $d^{(2)}(C_3 C_2 0)$ for site (130) would be the same as for site (133), but the term $d^{(3)}(C_3 C_2 C_1)$ disappears for site (130). On the other hand, the value –1 reflects the logic of accessibility that we apply: the distance $d_3(133)$ differs from $d_3(131)$ because of the fact that site (131) is located on the axis which links (130) to the center (100). It is then closer to the center (000) than sites (132) and (133).

For the order $k = 2$ one obtains:

$$\zeta^{(2)} = (-1)^{(\delta_{C2,1} - \delta_{C3,0})} (1 - \delta_{C2,0})(1 - \delta_{C2,0}\delta_{C1,0})$$

For $k = 1$, certain sites change their character: for example, the distance of (111), which is shortened in relation to (113) in order to reach site (100), is now longer to join (000). One obtains the following operator:

$$\zeta^{(1)} = (1 - \delta_{C1,0})(-1)^{\delta_{C1,1}(1-\delta_{C2,0}\delta_{C3,0}) + \delta_{C2,1}(1-\delta_{C3,0})(\delta_{C1,1}+\delta_{C1,3})}$$

in which we used the definition of the function δ_{ij}:

$$\delta_{i,j} = \begin{cases} 1\,\text{si}\ i = j \\ 0\,\text{si}\ i \neq j \end{cases}$$

Until now we considered only the distances from the residential areas to the centre of highest order $K = 3$. A similar formulation may be given for the other orders. E.g. the distance to reach the centers $K = 2$ reads:

$$d_2(C_3 C_2 C_1) \equiv d(C_3 C_2 C_1,\ C_3 00) = d^{(2)}(C_3 C_2 0) + d^{(1)}(C_3 C_2 C_1)$$

The general formula for the partial distances remains valid, but the formulas for the prefactors change. Indeed the partial distances can no longer be negative for $k = 2$, what yields:

$$\zeta^{(2)} = (1-\delta_{C2,0})(1-\delta_{C2,0}\delta_{C1,0})$$

For similar reasons the formula for ζ_1 becomes simpler, too:

$$\zeta^{(1)} = (1-\delta_{C1,0})(-1)^{\delta_{C1,1}(1-\delta_{C2,0})}$$

To reach centers of order K=1, simply one partial distance remains. The prefactor is then always +1 or 0:

$$\zeta^{(1)} = (1-\delta_{C1,0})$$

It is also possible to find a general formula for the indices $l(k)$, $m(k)$, $n(k)$ by introducing a new code $\widetilde{C}_i$ such as $\widetilde{C}_i = 1$ for all the values $C_i \neq 0$. With this binary coding, all the satellite sites are then equalized (this is possible because we have taken account of the reductions of the distances for certain sites with the value of the pre-factor ζ_k). By computing the differences $\Delta\widetilde{C}_i = \widetilde{C}_i - \widetilde{C}_i' \quad i = 1,2,3$, the exponents $l(k)$, $m(k)$, $n(k)$ of the partial distances $d^{(k)}(C_3C_2C_1)$, k = 1,2,3 are linked to the codes $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3$ and $\widetilde{C}_1', \widetilde{C}_2', \widetilde{C}_3'$ in the following manner:

$$l(3) = 0 \qquad m(3) = \Delta\widetilde{C}_3 \qquad n(3) = 2\Delta\widetilde{C}_3$$

$$l(2) = \widetilde{C}_3\Delta\widetilde{C}_2 \qquad m(2) = \Delta\widetilde{C}_2 \qquad n(2) = \Delta\widetilde{C}_2$$

$$l(1) = (\widetilde{C}_2 + \widetilde{C}_3)\Delta\widetilde{C}_1 \qquad m(1) = \Delta\widetilde{C}_1 \qquad n(1) = 0$$

This formulation is also valid if we consider the distances between sites (C_1, C_2, C_3) and the centers which belong to another hierarchical level such as (010) or (110). A similar formalization can be done for computing the distances to the rural areas, but the expression becomes more cumbersome. Both types of distance formulas do not allow for an analytical solution. In order to study its properties, we had to realize numerical simulations based on parameters, which were as close as possible to that of reality. This is the aim of section 2.

2.2 Microeconomic model of residential localization

To keep notation simple in the presentation of the economic model, we omit in this section the code of the considered site. The distances $d_k \equiv d_k(C_1C_2C_3)$ were discussed in the previous section. We also consider to the local amenities defined by d_0 = and e_0 = .

We assume that the household has an annual income W which is allocated:

- to renting a residential plot of size Z at a cost of $R + R_A$ (R_A is the agricultural opportunity rent while R is a differential rent).
- to commuting trips from each site towards the center of the city at a distance of d_K (with N being their annual amount).

- to the purchase of a quantity X of a composite good made of all the goods other than housing and travel; we assign the price of this composite good as the numeraire.

The household uses the balance of its income to obtain amenities and, urban and rural public goods which are differentiated at levels k. They are available free of charge in such a way that only the costs of the trips are borne by the residents. The urban amenities (the rural, respectively) are situated at distances of $d_0,\ldots, d_K$ ($e_0,\ldots,e_K$) and the annual amount of trips made to consume them is $a_0,\ldots, a_K$ ($b_0,\ldots,b_K$).

The unit cost of unitary transportation is t, no matter what the type of trip.

The household budgetary constraint can be written as:

$$W = (R_i + R_A)Z_i + tNd_{Ki} + X_i + t\sum_{k=0}^{K} d_{ki}a_{ki} + t\sum_{k=0}^{K} e_{ki}b_{ki}$$

in which i designates the urban site of the Sierpinski carpet where the household resides. We omit this index from the rest of this section because the microeconomic program is the same for all i. We call available income $W_D = W - tNd_K$ the gross-net income of the cost of alternating migration towards the job center.

Household preferences are expressed by means of a Cobb-Douglas function of utility, the attributes of which are the composite good, the residential plot, the urban amenities and, finally, the rural amenities. These amenities are imperfectly substitutable, which is expressed in a CES (*Constant Elasticity of Substitution*) sub-function where the parameter is $\rho < 1$ for urban amenities ($\sigma < 1$ respectively for rural amenities). This is an indicator of household taste for diversity. The function of utility is then a Cobb-Douglas / CES. That is, the level of utility offered by the urban amenities $A = \left(a_0^\rho + \ldots + a_K^\rho\right)^{1/\rho}$ and that which is offered by rural amenities $B = \left(b_0^\sigma + \ldots + b_K^\sigma\right)^{1/\sigma}$.

The household's problem consists in choosing a quantity of composite goods and the size of a residential plot, and to program trips for urban and rural amenities in such a way as to maximize their utility while respecting their budgetary constraints. This is written as:

$$\text{Max}_{X,a_1,\ldots,a_K,b_1,\ldots,b_K} U = \frac{1}{\alpha^\alpha \beta^\beta \gamma^\gamma \delta^\delta} X^\alpha Z^\beta \left(\sum_{k=0}^{K} a_k^\rho\right)^{\frac{\gamma}{\rho}} \left(\sum_{k=0}^{K} b_k^\sigma\right)^{\frac{\delta}{\sigma}} \qquad (1)$$

with the constraint:

$$W_D = W - tNd_K = X + (R_A + R)Z + t\sum_{k=0}^{K} d_k a_k + t\sum_{k=0}^{K} e_k b_k$$

In setting down, without loss of generalities, $\alpha + \beta + \gamma + \delta = 1$, $\alpha, \beta, \gamma, \delta > 0$.
In order to solve this problem, we use a two-step method suggested by Fujita *et al*. (1999) which as been used in the case of urban amenities, with the understanding that it is identical for rural amenities (by replacing d, a, D, A by e, b, E and B). We primarily look for the program of travel for urban amenities which enables the household to reach a given level of utility A at the least cost:

$$\text{Min}_{a_0,\ldots,a_K} \sum_{k=0}^{K} d_k a_k \quad \text{with the constraint:} \quad \left(a_0^\rho + \ldots + a_K^\rho\right)^{1/\rho} = A \qquad (2)$$

The resolution of this optimization program with the constraint of the Lagrangian method gives:

$$A = a_0 \left[\sum_{k=0}^{K} \left(\frac{d_k}{d_0} \right)^{\frac{\rho}{\rho-1}} \right]^{\frac{1}{\rho}} = a_0 \left(\frac{D}{d_0} \right)^{\frac{1}{\rho-1}} \qquad \text{with: } D = \left[\sum_{k=0}^{K} d_k^{\mu} \right]^{\frac{1}{\mu}} \qquad \text{and: } \mu = \frac{\rho}{\rho-1} \tag{3}$$

The coefficient D can be interpreted as the inverse of the global index of the accessibility of urban amenities based on a given residential site. We thus obtain a_k as a function of A:

$$a_k = \left(\frac{d_k}{D} \right)^{\frac{1}{\rho-1}} A \qquad k = 0, \ldots, K \tag{4}$$

We express the cost of travel for urban amenities as a function of the level of utility which is associated with the latter by substituting in (2):

$$\sum_{k=0}^{K} d_k a_k = DA$$

As a result, the household problem is:

$$\text{Max}_{X,A,B} U = \frac{1}{\alpha^{\alpha} \beta^{\beta} \gamma^{\gamma} \delta^{\delta}} X^{\alpha} Z^{\beta} A^{\gamma} B^{\delta} \ \text{ s/c: } \ X + (R_A + R)Z + tDA + tEB = W_D \tag{5}$$

One obtains the conditions of the usual first order by invalidating the partial derivatives of Lagrange's multiplier:

$$X^* = \alpha W_D, \quad Z^* = \frac{\beta W_D}{R + R_A}, \quad A^* = \frac{\gamma W_D}{tD}, \quad B^* = \frac{\delta W_D}{tE} \tag{6}$$

Consequently, by substituting A^* in (4) (idem for B^*), we have the optimal number of household trips towards the sources of amenities of different types. The optimal level of utility is U^*:

$$U^* = \frac{1}{(R + R_A)^{\beta} t^{\gamma+\delta} D^{\gamma} E^{\delta}} W_D \tag{7}$$

At equilibrium, the same level of utility is reached by each household no matter what its residential location. Moreover, we assume an open city situation in which this level of utility at equilibrium is given by its value in the rest of the world: $U^* = \overline{U}$. We can then determine the amount of available income in an area i at equilibrium and, in this way, the land rent in this area:

$$R = R_A + \left(\frac{W - tNd_K}{\overline{U} t^{\gamma+\delta} D^{\gamma} E^{\delta}} \right)^{\frac{1}{\beta}} \tag{8}$$

The optimal size of a plot is given by:

$$Z = \frac{\beta (W - tNd_K)}{R_A + \left(\frac{W - tNd_K}{\overline{U} t^{\gamma+\delta} D^{\gamma} E^{\delta}} \right)^{\frac{1}{\beta}}} \tag{9}$$

Population Q is equal to the inverse of the size of the plot multiplied by the size of the residential site.

Based on available household income, we obtain optimal decisions by (4), (6), (8) and (9) in a context of urban equilibrium as a function of the parameters of the problem (W, t, $\overline{U}$, R_A, N, α, β, γ, δ, ρ, σ).

3. Simulations: a benchmark

3.1 Method

We are here interested in three outputs of the model: the land rent [8], the size of the residential plots [9] and the population of the metropolitan area. This latter is equal to the inverse of the size of the residential plots, multiplied by the area of the residential sites (with some adjustments which will be given later). In section 1, we developed an open city model, where migrations are possible with the rest of the world. Nevertheless, a closed city model is more realistic, that is to say an urban area for which population Q is given and where the utility level reached by the residents is endogenous. In this case, one derives from [9]

$$Q = 2\pi \int_{-f}^{f} \frac{S_i}{Z_i} di \qquad [10]$$

where $\pm f$ are the limits of the urban area, and S_i is the size of the residential site i, as determined by the model.

Land rent is given by (8) and depends on the utility of the rest of the world, which is unobservable and must be endogenous in a closed city model. It is here expressed as a function of the land rent at the CBD, noted $R(0)$ and obtained for the households located in site (000):

$$U = \frac{W}{t^{\gamma+\delta} D(0)^{\gamma} E(0)^{\delta} \left[R(0) + R_A\right]^{\beta}}, \qquad (11)$$

where $R(0)$ and R_A are observable. By using (11) in (8) and (9), we obtain the equations of land rent and of the residential plots size which, together with (10), are the results used in our simulations.

In the real world, the land rent $R^o(x)$ and the size of the residential plots $Z^o(x)$ are observed according to the distance from the center of the urban area (superscript o is used for observed values, x is the distance from the center of the urban area). The limits of the metropolitan area, $\pm f^o$ and its population Q^o are also known. We here choose parameters for the theoretical model in such a way that the predicted values noted $R^p(x)$, $Z^p(x)$, f^p and Q^p are close to the observed values. Some parameters are known and available such as household income W; others are more difficult to get and need some computation (for instance: the generalized transportation cost t) or experts' opinion (for instance: R_A). Finally, some other parameters are unobservable, such as those characterizing the preferences (β, γ, δ, etc.).

3.2. Parameters

The French Housing Survey carried out in 1996 by the INSEE (National Institute of Statistics and Economic Studies) contains most of the variables necessary for estimating/computing the parameters included in our model. We select 23 French urban areas with a city-center containing 100 to 200,000 inhabitants[4,5]. On the average, a city-centre is surrounded by (1)

[4] A French "urban area" is a densely built urban 'pole' (city-center + suburbs) offering more than 5,000 jobs, and a 'periurban belt' made up of communes which are not adjacent to the pole (separated by agriculture, forests, etc.), from which 40% or more of the active population commutes outside of the commune, generally to the urban pole.

[5] Our scope of investigation is limited to detached houses that are occupied by the owner; this corresponds to 1,706 observations in the Housing Survey. We have indeed eliminated (1) apartment buildings because the theoretical model applies to residential plots, and (2) rented housing (not numerous among the detached houses) since their rent is not commensurate with the price of purchasing a house.

15.3 suburban "communes" (the commune is the French lowest administrative level) containing 8200 inhabitants (median: 5200), and (2) a periurban belt made of 69,000 inhabitants spread over 74 communes (on the average: 930 inhabitants/commune; median: 550). Hence, each urban area includes - on the average - 90 communes, that is to say a number quite close to the 125 sites of the Sierpinski carpet with K=3. This explains our choice.

However, the correspondence between the "communes" and the residential sites of a Sierpinski carpet requires some further adjustments. We have defined the 5 generator sites as the city-center of the urban area, the 20 sites which result from iteration 2 as the corresponding 15.3 suburban communes, and the 100 sites from step 3 as the 74 communes of the periurban belt. The unit of distance was chosen in relation to the size of the selected urban areas, by setting distance d(133,000) equal to 24 kilometers. The surface unit is deduced from the distance definition, especially for the residential sites. A proportion of the area of the residential sites is not available for housing: it is dedicated to road networks and economic activities, which represent 40% of the urban space (this rate is based on national data on land use in France).

The other parameters used in the model are also derived from the 1996 French Housing Survey (further information about the computations is available upon request to the authors):

- Household income: W = 30 300 €/household/year.
- Actualization rate used to annualize the cost of buying a house: 5%.
- Urban land rent at the origin $R(0)$ = 5,06 €/m²/year (see Appendix 1).
- Agricultural rent R_A = 0,015 €/m²/year (Source: expert's opinion).
- Generalized unitary transportation cost (t) is the sum of the direct monetary cost (computed on the basis of the French Fiscal Administration at 0,30 €/km in 1996) and the opportunity cost of time which is evaluated by experts at 0,15 €/min. On the basis of a round-trip and a speed computed from the Housing Survey, we obtain an annual generalized transportation cost of 0.9 €/km.
- We estimate that a worker commutes for work 200 times per year; this computation assumes 5 weeks of paid holidays, 5 working days per week, 10 days of public holidays and 25 days that were not worked (35 hour workweek, part-time, flextime, sick leaves, maternity leaves, etc.). We consider that a household is composed of 1.5 workers. Hence, N = 300.
- All members of a household travel together to urban/green amenities (one single trip by household and visit).

As for the unobservable parameters which characterize preferences, the following values have been chosen. β is the proportion of available income devoted to residential consumption free of the commuting cost; it is here to 0.20[6]. We have selected identical initial values for the parameters which characterize the taste in urban and green amenities: γ= 0,065 and δ= 0,065 (these values vary in the static comparative simulations). Substitutability between amenities are fixed by ρ = –30 and σ = –30; these values are a compromise between complementarity and unitary substitutability. The multifractal parameter is equal to 0,50, which means that

[6] The French Housing Survey indicates that, for households that have recently become landowners, reimbursing the loan represents 22% of the gross income (or about 25% of the income net of commuting). Nevertheless, the theoretical model takes into account the land rent, equal to the price of the land, whereas the building itself is included in the composite a-spatial good. The price of land represents roughly 28.4% of the total cost, which results in a value for β close to 10%, which is a value superior that those generally found in the literature. Hence, β is fixed to 20%.

during the first iteration, the central site represents 50% of the built-up area while each of the four peripheral sites represents 12,5%.

3.3. Results

Figures 5, 6 and 7 give respectively the spatial distribution of the size of the residential plots, the land rent and the population[7] for these parameters; we here restrict the graphical representation to the north-east quadrant. The expected gradients of an increase in plot size, a decrease in land rent, and in a reduction in population according to distance from the CBD are observed. The size of the residential plots is 1390 m^2 in the central site and 3170 m^2 in the periphery. However, the progression is not monotonous: for instance, the sites located on the principal diagonal are poorly located both in relation to the urban public goods of the central site and to the peripheral green amenities; hence, their inhabitants must receive large residential plots to compensate for this poor accessibility. Obviously, land rent has an inverse evolution from that of residential plots.

Figure 5. Lot size

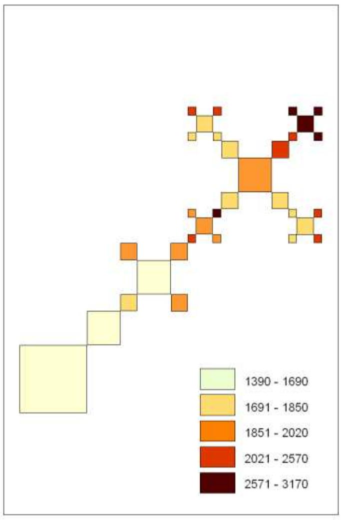


Figure 6. Land rent

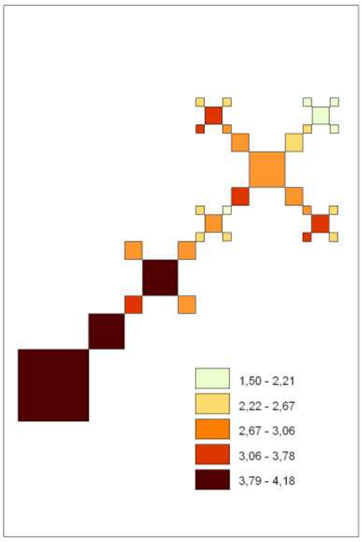
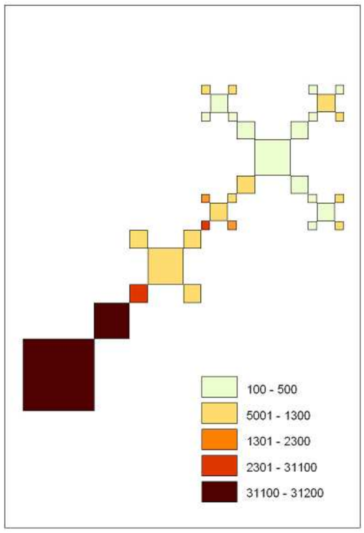


Figure 7. Population

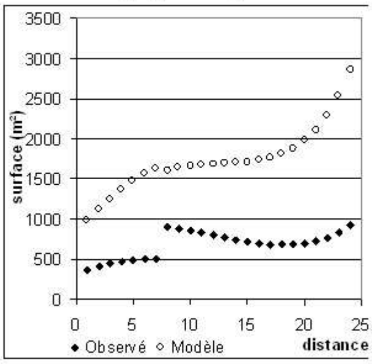


In the less populated sites, population consists of about one hundred inhabitants, due to both the small size of the sites owing to the multifractal dimension of the carpet and the large size of their residential plots. The population consists of 31 200 inhabitants in the central site (Let us remind that we here restrict ourselves to households living in detached houses; the inhabitants of apartment buildings do not consume residential land surface).

It is difficult to directly compare predicted values (obtained from 125 site values) and observed values (obtained from econometric regressions). Therefore, we also regress the predicted values; the regressors are distance (Euclidean distances, third degree polynomial) and dummy variables which identify the commune-center and the suburban communes (if significant). The two situations are compared in figures 8, 9 and 10.

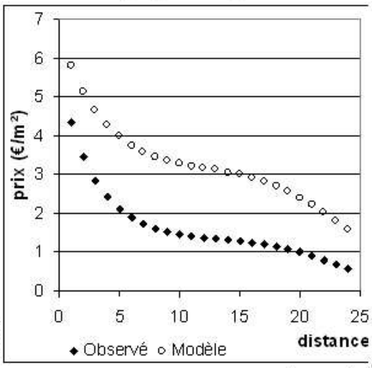
[7] We are aware that population should strictly be mapped by a symbol map; it was here replaced by a choropleth map for a better and easier comparison.

Figure 8. Benchmark simulation: Size of plots according to distance (regression)



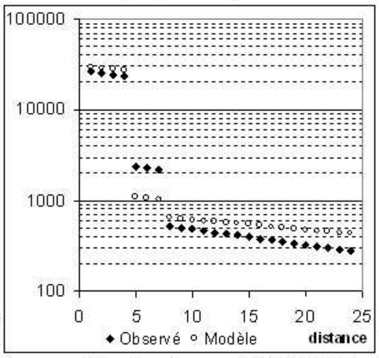
Sources: Housing Survey 1996 (INSEE) and model

Figure 9. Benchmark simulation: Land rent according to distance (regression)



Sources: Housing Survey 1996 (INSEE) and model

Figure 10. Benchmark simulation: Population according to distance (regression)



Sources: Housing Survey 1996 (INSEE) and model

The residential plots derived from the model are noticeably larger than those observed in the real world; land rent is also higher in the model, but by a smaller proportion. The population of the model is too large in the commune-center, weak in the suburban communes and again higher than the actual value in the periurban communes. Keeping in mind the high level of abstraction of the model, the magnitude order seems to be sufficiently acceptable to realize some comparative static simulations by varying the parameters of household behavior or those which characterize the economy. This is the objective of the following section.

4. Comparative static

It is common use to consider that the main causes of urban spread are the increase in population as well as in income, and the decrease in transportation costs. However, for several decades, this urban extension has taken on a particular pattern. In 1980, the American House of Representatives said that it was « like Swiss cheeses with more holes than cheese » (cited by Burchfield *et al.*, 2006). As a result, we are here interested in the factors which may explain the discontinuous development, or "leap frogging": numerous authors have added a change in household preferences for a "green" living environment to the three preceding factors which would contribute to explain these new forms of metropolization (see e.g. Brueckner *et al.*, 1999). We here examine how the benchmark situation that we developed in the preceding section evolves under the effect of these factors.

4.1. Changes in preferences

Figures 11 and 12 illustrate the variation in the size of the residential plots (in square meters) when the taste for green amenities is larger than that for urban public goods ($\gamma = 0.03$ and $\delta = 0.1$) and inversely ($\gamma = 0.1$ and $\delta = 0.03$), while these parameters were equal in the benchmark simulation. Land rent is here not represented because the pattern is almost the inverse of the figure obtained for the size of residential plots.

On the one hand, when the taste for green amenities is greater than that for urban public goods (Figure 11), the size of the residential plots is often smaller than that of the benchmark, particularly in the periphery of the Sierpinski carpet, thus marking a substitution between residential consumption and the consumption of green amenities. Nevertheless, one observes an increase in the size of the residential plots for most of the sites located on the principal diagonal, particularly in the center of the carpet: here, poor accessibility to green amenities must be compensated by larger parcels. The changes obtained for the population (which are not illustrated here) show a movement from the center towards the periphery of the Sierpinski carpet.

On the other hand, when urban amenities are preferred to green amenities (Figure 12), households living in the periphery must have larger residential plots which compensate for their long trips to urban sites, including the center of the carpet. When households strongly taste urban amenities, the tropism of the central site (site 000) is strong: even in ~~the~~ urban centers at a high level in the urban hierarchy, such as those obtained by the following iteration (as site 100) (cf. the codes in figure 3) the size of the residential plots decreases, because they are not enough central.

We have also analyzed the effects of changes in the substitutability between green amenities (as well as urban amenities). In both cases, the plot size decreases when substitutability between green (or urban) amenities increases: households find green (or urban) amenities close at hand, and they consume these in place of other green (or urban) amenities that are more distant. This indicates that the more the taste for variety is pronounced, the more a geometric pattern such as the Sierpinski carpet improves the welfare of the inhabitants, by offering a variety of urban and green sites because of the interlocking scale.

Figure 11. Simulation: Increased preference for green amenities. Plot size

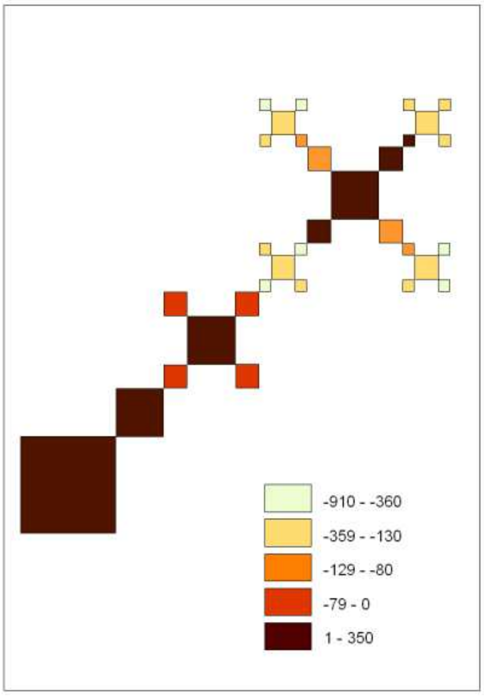
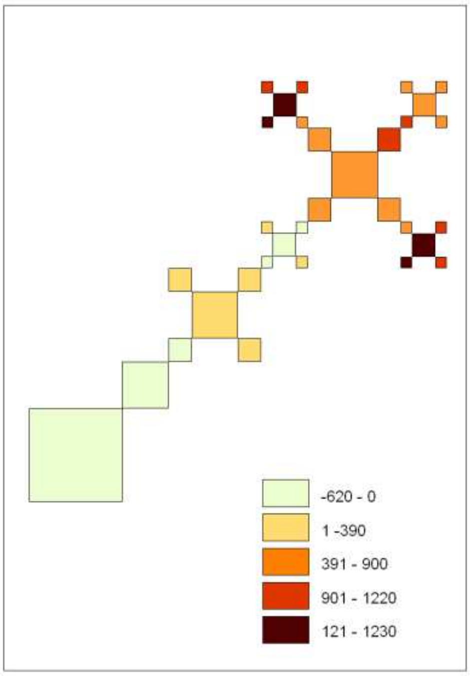


Figure 12. Simulation: Increased preference for urban public goods. Plot size



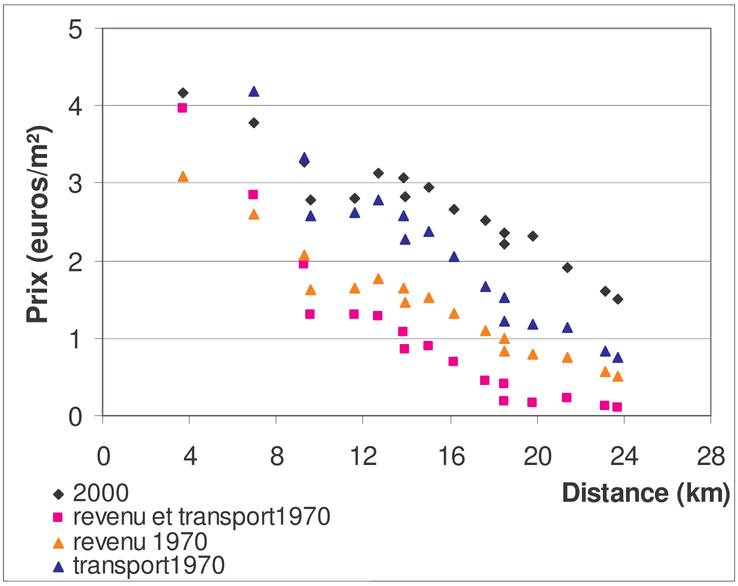
4.2. Increase in income and decrease in transportation cost

Let us now consider a situation where the income by inhabitant is lower than it is today but the transportation cost higher. We know that individual income in France increased by 60% between 1970 and 2000 (adjusted for inflation), and transportation cost decreased; we fix this latter rate of decrease at 60% for the simulations[8]. We then reconstruct the situation in the early 1970s by examining the effects of these two evolutions, separately or simultaneously. Figure 13 shows these effects on land rent.

The two evolutions taken together show that land rent in 1970 was almost equal to zero at less than roughly 20 kilometers from the CBD, whereas it is about 1.5 € per square meter at 24 kilometers in the benchmark (2000). A low rent in 1970 results in very small population (near zero) and in huge residential plots: obviously, the front line of periurbanization did not at this time come within twenty kilometers. The economic evolutions in income and transportation cost then seem to have played an important role in the urban spread which characterized the end of the last century.

The effects of income and transportation cost are not the same according to distance, as shown in Figure 13. If income remains constant and transportation cost decreases, land rent decreases by somewhat similar proportions no matter the distance. In the opposite case, if only income evolves, land rent is little affected for up to twelve kilometers and it decreases sharply beyond that. It is well known (Wheaton, 1974), as is suggested by intuition, that the decrease in transportation costs entails an increase in land rent in the periphery of urban areas. However, its effect is smaller than that of income between 1970 and 2000.

Figure 13. Simulation: Effect of the evolution of income and transportation costs on land rent



In all, it appears that the periubanization movement that France has witnessed for roughly thirty years can be explained by the combination of two economic factors: the increase in income and the decrease in the cost of automobile transportation, without reference to changes in preferences. If these occurred as well in favor for the green, their effects can only be added to the two previous ones.

4.3. New rise in income and decrease in transportation costs

Let us now simulate a new increase in income (from 30,300 to 40,000 €) and/or a new decrease in the unit transportation cost (from 0.9 to 0.5 €/km/year). Results are reported in Figures 14 and 15.

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[8] According to the INSEE price index and deduction made for the general increase in prices, in 1975, a car cost half as much as it does today. If the relative price of gasoline has recently increased (after the impacts of the oil slump which just recently brought it to the level of the 1960s), the speed of travel has also increased, particularly because of improvements in the road network, which reduces the overall transportation cost.

Simulations: Effects of an increase in income and/or a decrease in transportation costs …

Figure 14. … On the plot size

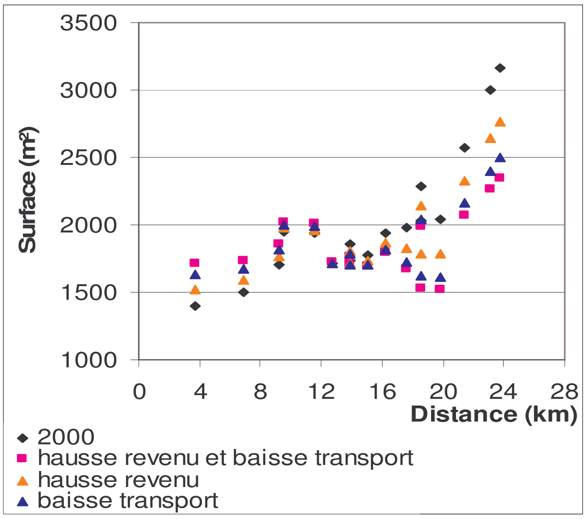
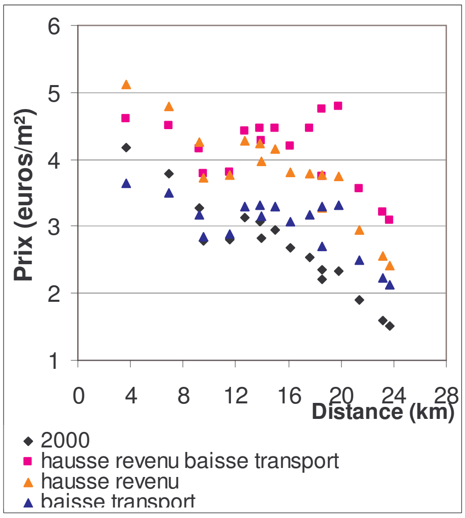


Figure 15. . … On land rent



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These effects on the size of the residential plots are in accordance to our expectations: the size increases in the central sites and decreases in the periphery. However, these evolutions are smaller than those previously simulated. Land rent flattens, increasing everywhere but particularly in the sites at some distance from the center: the rent is slightly higher to that of the center at a distance of about 20 kilometers. The effect on the population cannot be shown as it is very small; one only observes a small migration towards the periphery.

5. Conclusions

Economists and geographers often underline the heterogeneity of metropolitan areas, which breaks away from the homogeneity of the interlocking Thünian rings, and which is made up of interlacing built-up sites and recreational/agricultural open spaces. Economics poorly integrates these urban patterns into models because heterogeneity is difficult to model in a two-dimensional space (see e.g. Ogawa and Fujita, 1982; Lucas and Rossi-Hansberg, 2002). Geography is often more able to represent these heterogeneous spaces which make up the core of the discipline, but often without any economic or sociological theoretical background, even if the challenge is here to explain how human action creates these spaces. Anas *et al.* (1998) has already stressed this division between the two disciplines; nevertheless both are necessary and complementary to explain urban structures. The present paper tries to fill a gap in this direction.

We here used an urban microeconomics model of residential localization, with the geometry (and hence the geography!) of a multifractal Sierpinski carpet. From an economic point of view, households consume differentiated urban public goods and green amenities, with a preference for diversified baskets. The variety of amenities is the result of urban sites, which offer urban goods ranked according to the Sierpinski carpet hierarchy; the green lacunas separating urban areas are also hierarchically structured. The economic model leads to an analytical solution by using a coding system for the sites of the Sierpinski carpet, which allows the computation of the distances between any two sites.

In order to study the properties of this model, we used a set of parameters derived from the observed statistical reality of the French medium sized urban areas. We have simulated retrospective or prospective evolutions starting with a benchmark situation. Findings show that an increase in preferences for green amenities flattens land rent, residential plot sizes and population gradients, and leads to an overall extension of the metropolitan areas towards more outlying residential locations. Moreover, when the substitutability between either urban or green amenities decreases, some complex patterns (such as the level of interlocking on the scale of the multifractal Sierpinski carpet) are suitable because they offer to the households a variety of differentiated public goods close to their residence.

Regarding the economic aspects, it appears that, in a country like France, the strong increase in income over the last thirty years combined with a decrease in the transportation cost is enough to explain the further extension of periurbanization, independently of possible changes in preferences. By further simulating in same direction new evolutions in income and/or transportation costs, effects will not be of the same extent in terms of metropolitan sprawl/spread.

The analyses performed in this paper really illustrate the interest of approaches combining economics and geography for representing spaces formed by heterogeneous objects. However, the level of abstraction is much too high to allow applications for real world urban patterns. The possible extensions follow the line of a reduction in distance between the abstract world of models and the real world. It is possible to combine several fractals (Sierpinski carpet, teragons and Fournier dusts) in order to do so; this would allow more complex geometric representations while maintaining their theoretical intelligibility or even to realize an econometric calibration based on a richer set of equations than the ones we used. It is also conceivable to define the rules of economic behavior which can generate fractal patterns by other methods (cellular automata, agent-based modeling) rather than beginning with geometry a priori.
This paper is a first attempt is this direction

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APPENDIX 1
ESTIMATING THE LAND RENT FUNCTION

We estimate by regression, the logarithm of the housing price (real estate price) for households who recently purchased a house in a French medium-sized city in 1996 (Source: Housing Survey; 292 observations available). The independent variables are the living space in the purchased house and the area of the garden (the sum of which is Z, the size of the residential plot in the model), the distance to the CBD of the urban area ("commune centre"), the age of the building, the number of bathrooms, a dummy variable indicating if the commune belongs (1) or not (0) to the Mediterranean region as well as the fiscal income of the commune (see Cavailhès, 2005 for more details).

Hausman's test shows that only the living space is endogenous. We use the 2SLS method with instrumental variables as estimation technique. Instruments are here characteristics of the household and characteristics of the location; they are verified to be exogenous by the test of Sargan. The following results are obtained:

Variable	Parameter	Student *t*
Constant	8.03030	17.73
Living space (m²)	0.96448	9.53
Number of bathrooms	0.14404	6.44
Building construction: ≥ 1990	0.39303	6.10
1982-89	0.27416	3.91
1975-81	0.29466	3.53
1968-74	0.27461	2.58
1962-67	0.49101	5.19
1949-61	0.03585	0.40
1915-48	0.04277	0.61
Surface of the garden (m²)	9.354 E-5	4.02
Average communal income	0.00308	2.74
Mediterranean region	0.10190	1.94
Distance from the center	-0.08523	-4.13
Distance from the center (sq)	0.00636	2.47
Distance from the center (cubic)	-1.618 E-4	-1.92

The total predicted real estate value is computed at the average point of the independent variables other than distance, and it is annualized at a rate of 5%. The obtained value at the origin is 17.8€/m²/year. Regarding the 118 observations for which the plot of land and the building were purchased separately, we compute that the price corresponds to 28.4% of the total cost of housing. Land rent at the CBD (noted $R(0)$) is then 5.06€/m²/year. The parameters of distance are used in the computation of the error formula.